Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (mm 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Sheffer stroke KU-algebras

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Received 5 August 2025; Revised 18 August 2025; Accepted 10 September 2025

Abstract. In this paper, we introduce the concept of $SSKU$-ideals of a $SSKU$-algebra, and study some of its properties. Also, we discuss some properties of the image [resp. preimage] of $SSKU$-ideals under a $SSKU$-homomorphism.

2020 AMS Classification: 06F05, 03G25, 03G10

Keywords: $SSKU$-algebra, $SSKU$-homomomorphism, SKU-ideal, The image and preimage of KU-ideal, .

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1. Introduction

The class of logical algebras including BCK-algebras (Iséki and Tanaka [1]), BCI-algebras (Iséki [2]), BCH-algebras (Hu and Li [3]), BCC-algebras (Dudek [4]), QS-algebras (Ahn and Kim [5]), Q-algebras (Negger et al. [6]), BE-algebras (Kim and Kim [7]), and UP-algebras (Iampan [8]) are collectively referred to as Boolean-like algebras and serve as a foundational framework in digital circuit design and theoretical computer science.

In particular, Prabpayak and Leerawat [9] introduced a new algebraic structure called a *KU-algebra* and investigated some of its properties. Further developments on KU-algebras can be found in [10, 11, 12, 13, 14, 15, 16, 17]. More recently, in 2024, Ansari and Koam [18] examined several properties of modules over KU-algebras. Additionally, Beak et al. [19] introduced the notion of Γ-KU-algebras as a generalization of KU-algebras and explored various structural aspects.

On another front, Sheffer [20] first introduced an operation known as the Sheffer stroke on a set X (see also [21, 22]). Utilizing the Sheffer stroke, Oner et al. [23] defined and analyzed Sheffer stroke-commutative, positive, and implicative-positive BCK-algebras, establishing interrelations among them (see [24, 25, 26, 27, 28]). In particular, Oner and Katican [29] introduced the concept of a Sheffer stroke UP-algebra and studied homomorphisms between such structures.

In the study of logical algebras, the concept of an ideal plays a crucial role in analyzing the structure and inherent properties of these algebras. Ideals contribute to the simplification and optimization of digital circuits, facilitate structural analysis, ensure logical consistency, and support a wide range of applications in computer science and cryptography. Motivated by this, we aim to investigate the structure of ideals in Sheffer stroke KU-algebras defined over a groupoid X.

The organization of this paper is as follows.

In Section 2, we recall some basic definitions and preliminary concepts that will be used throughout the paper. In Section 3, we introduce the notion of an $SSKU$-ideal in a $SSKU$-algebra and explore its fundamental properties. Section 4 is devoted to studying the behavior of $SSKU$-ideals under $SSKU$-homomorphisms between KU-algebras. In particular, we examine the images and preimages of $SSKU$-ideals under such homomorphisms and analyze structural aspects related to the kernel $\ker f$ of a given homomorphism f.

2. Preliminaries

In this section, we recall basic fundamental concepts of a KU-algebra and a Sheffer stroke operation, and give an Example .

Definition 2.1 ([9]). An algebra $(X, *, 0)$ of type $(2, 0)$ with a binary operation $*$ is called a *KU-algebra*, if it satisfies the following conditions: for every $x,\ y,\ z \in X$,

(KU_1) $(x * y) * [(y * z) * (x * z)] = 0$,

(KU_2) $x * 0 = 0$,

(KU_3) $0 * x = x$,

(KU_4) $x * y = 0 = y * x$ implies $x = y$.

Remark 2.2 ([9]). (1) From (KU_1) and (KU_2), we have

$$x * x = 0 \text{ for each } x \in X. \tag{2.1}$$

(2) From (KU_1), (KU_2) and (KU_3), we get

$$z * (x * z) = 0 \text{ for every } x,\ z \in X. \tag{2.2}$$

We define a binary relation $\leq$ on X as follows: for any $x,\ y \in X$,

$$x \leq y \text{ if and only if } y * x = 0.$$

Result 2.3 (See [17]). *An algebra $(X, *, 0)$ is a KU-algebra if and only if it satisfies the following conditions: for all $x,\ y,\ z \in X$,*

($\mathrm{KU}_{1'}$) $(y * z) * (x * z) \leq x * y$,

($\mathrm{KU}_{2'}$) $0 \leq x$,

($\mathrm{KU}_{3'}$) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

($\mathrm{KU}_{4'}$) $x \leq y$ *if and only if* $y * x = 0$.

Result 2.4 (See Lemmas 2.5, 2.6 and 2.7, [17]). *In X, the followings hold: for all $x,\ y,\ z \in X$,*

(1) $x \leq y$ *implies* $y * z \leq x * z$,

(2) $z * (y * x) = y * (z * x)$,

(3) $y * [(y * x) * x] = 0$.

Definition 2.5 ([30]). A subset S of X is called a *subalgebra* of X, if $x * y \in S$ for every $x,\ y \in S$.

Definition 2.6 ([30]). Let I be a nonempty subset of X. Then I is called a *KU-ideal* of X, if it satisfies the following conditions: for every $x,\ y,\ z \in X$,
(KUI_1) $0 \in I$,
(KUI_2) $x * y,\ x \in I$ implies $y \in I$.

In a KU-algebra X, we define $x \dot{\wedge} y = (y * x) * x$ for all $x,\ y \in X$.

Definition 2.7. A KU-algebra X is said to be:
(i) *KU-commutative* [31], if it satisfies the following condition:

$$(y * x) * x = (x * y) * y, \text{ i.e., } x \dot{\wedge} y = y \dot{\wedge} x \text{ for all } x,\ y \in X, \tag{2.3}$$

(ii) *bounded* [32], if there is the greatest element 1 (called the *unit*) of X and $x * 1$ is denoted by N_x for each $x \in X$.

We now proceed to define the Sheffer stroke operation on the set X, originally introduced by Chajda [21].

Definition 2.8 ([21]). Let $(X, \circ)$ be a groupoid. Then $\circ$ is said to be *Sheffer stroke operation* on X, if it satisfies the following conditions: for all $x,\ y,\ z \in X$,
(S_1) (Commutativity) $x \circ y = y \circ x$,
(S_2) (Absorption) $(x \circ x) \circ (x \circ y) = x$,
(S_3) $x \circ [(y \circ z) \circ (y \circ z)] = [(x \circ y) \circ (x \circ y)] \circ z$,
(S_4) (Absorption) $[x \circ ((x \circ x) \circ (y \circ y))] \circ [x \circ ((x \circ x) \circ (y \circ y))] = x$.

From (S_2), it is obvious that

$$(x \circ x) \circ (x \circ x) = x \text{ for each } x \in X. \tag{2.4}$$

Result 2.9 (Lemma 1, [21]). *Let $(X, \circ)$ be a groupoid with a Sheffer stroke. We define a binary relation $\leq$ on X as follows: for all $x,\ y \in X$,*

$$x \leq y \text{ if and only if } x \circ y = x \circ x. \tag{2.5}$$

Then $\leq$ is a partial order on X.
In this case, $\leq$ is called the *induced order* on X.

Result 2.10 (Lemma 2, [21]). *Let $(X, \circ)$ be a groupoid with a Sheffer stroke and $\leq$ the induced order of X. Then for every $a,\ x,\ y \in X$,*
(1) *$x \leq y$ if and only if $y \circ y = x \circ x$,*
(2) *$x \circ [y \circ (x \circ x)] = x \circ x$ is the identity of X,*
(3) *$x \leq y$ implies $y \circ z \leq x \circ z$,*
(4) *$a \leq x$ and $a \leq y$ imply $x \circ y \leq a \circ a$.*

Definition 2.11 ([21]). A groupoid $(X, \circ)$ is called a *Sheffer stoke semigroup*, if it satisfies the conditions (S_1)–(S_4) and the following condition: for all $x,\ y,\ z \in X$,
(S_5) (Associativity) $x \circ (y \circ z) = (x \circ y) \circ z$.

Example 2.12. (1) Let $X = \{0, 1\}$, and $Y = \{0, 1, 2\}$ with the operations $\circ$ on X and Y, respectively. The corresponding Cayley tables for these structures are therefore constructed as follows. Then we can easily check that each $(X, \circ)$ is a Sheffer stroke semigroup.

$\circ$	0	1
0	0	0
1	0	1

$\circ$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Table 2.1

(2) Consider the Sheffer stroke groupoid $(X, \circ)$ with the following Cayley table: Then clearly,

$\circ$	0	1
0	1	1
1	1	0

Table 2.2

$$(0 \circ 1) \circ 1 = 0 \neq 1 = 0 \circ (1 \circ 1).$$

Thus $(X, \circ)$ is not a Sheffer stroke semigroup.

3. Sheffer stroke KU-algebras

We define a Sheffer stroke KU-algebra in this section and investigate some of its fundamental properties, accompanied by examples.

Definition 3.1. Let $\circ$ be a Sheffer stroke operation on a set X. Then a *Sheffer stroke KU-algebra* (briefly, $SSKU$-algebra) is a structure $(X, \circ, 0)$ of type $(2, 0)$ with the constant $0 \in X$ satisfying the following axioms hold: for all $x, y, z \in X$,

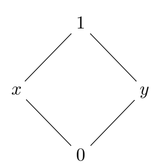
($SSKU_1$) $[((z\circ(x\circ x))\circ(z\circ(x\circ x)))\circ(((z\circ(y\circ y))\circ(y\circ(x\circ x)))\circ((z\circ(y\circ y))\circ(y\circ(x\circ x))))]\circ[((z\circ(x\circ x))\circ(z\circ(x\circ x)))\circ(((z\circ(y\circ y))\circ(y\circ(x\circ x)))\circ((z\circ(y\circ y))\circ(y\circ(x\circ x))))] = 0$,

($SSKU_2$) $x \circ x = x \circ (0 \circ 0)$,

($SSKU_3$) $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0 = (y \circ (x \circ x)) \circ (y \circ (x \circ x))$, imply $x = y$.

In particular, A SS-algebra X satisfying the condition (S_5) is called a *associative SS-algebra*.

Example 3.2. (1) (See Example 3.1, [23]) Let $X = \{0, x, y, 1\}$ be the set with the following Hasse diagram:



Consider the binary operation $\circ$ on X with the following Cayley table:

$\circ$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Table 3.1

Then we can easily check that $(X, \circ, 0)$ is an $SSKU$-algebra.
(2) Consider the Cayley table for a binary operation $\circ$ on $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	3	3	3	3
1	3	2	3	2
2	3	3	1	1
3	3	2	1	0

Table 3.2

Then $(X, \circ, 0)$ is an $SSKU$-algebra
(3) Consider the Cayley tables for a binary operation $\circ$ on $X = \{0, 1, 2, 3, 4, 5\}$:

$\circ$	0	1	2	3	4	5
0	3	3	3	3	5	5
1	3	2	3	2	3	2
2	3	3	1	1	1	2
3	3	2	1	0	0	0
4	5	3	1	0	4	4
5	5	2	2	0	4	0

Table 3.3

Then we can easily check that $(X, \circ, 0)$ is an $SSKU$-algebra.

Remark 3.3. The axioms (SSKU_1), (SSKU_2) and ($\mathrm{KU}SS_3$) are independent of each other (See Example 3.4).

Example 3.4. (1) Consider the Cayley table for a binary operation $\circ$ defined on the set $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	0	2	0	0
1	2	0	3	1
2	0	3	0	2
3	0	1	2	0

Table 3.4

Then we can easily see that the axiom (SSKU_1) holds. on the other hand, we have

$$1 \circ 1 = 0 \neq 2 = (1 \circ (0 \circ 0)),$$

$$(1 \circ (2 \circ 2)) \circ (1 \circ (2 \circ 2)) = 0 = (2 \circ (1 \circ 1)) \circ (2 \circ (1 \circ 1)) \text{ but } 1 \neq 2.$$

Thus the axioms (SSKU_2) and ($\mathrm{KU}SS_3$) do not hold.
(2) Consider the Cayley table for a binary operation $\circ$ defined on the set $X = \{0, 1, 2, 3\}$:
Then we can easily check that the axiom (SSKU_2) is satisfied but the axioms (SSKU_1) and ($\mathrm{KU}SS_3$) are not satisfied.

$\circ$	0	1	2	3
0	1	1	0	3
1	2	3	0	1
2	3	0	0	2
3	2	1	1	1

Table 3.5

$\circ$	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

Table 3.6

(3) Consider the Cayley table for a binary operation $\circ$ defined on the set $X = \{0, 1, 2, 3\}$:
Then we can easily see that the axiom (KUSS_3)holds but the axioms (SSKU$_1$) and (KUSS_2) do not hold.

(4) Consider the Cayley table for a binary operation $\circ$ defined on the set $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	0	3
2	0	0	0	0
3	0	3	0	0

Table 3.7

Then we can easily check the axioms (SSKU$_1$) and (KUSS_2) hold. On the other hand, we get

$$(1 \circ (0 \circ 0)) \circ (1 \circ (0 \circ 0)) = 0 = (0 \circ (1 \circ 1)) \circ (0 \circ (1 \circ 1)) \text{ but } 0 \neq 1.$$

Thus the axiom (SSKU$_3$) does not hold.

(5) Consider the Cayley table for a binary operation $\circ$ defined on the set $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	2	2	2
3	0	3	2	3

Table 3.8

Then we can easily see the axioms (SSKU$_1$) and (KUSS_3) hold. On the other hand, we have

$$1 \circ 1 = 1 \neq 0 = 1 \circ (0 \circ 0).$$

Thus the axiom ($SSKU_2$) does not hold.

(6) Consider the Cayley table for a binary operation $\circ$ defined on the set $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	0	1	0	0
1	1	1	1	1
2	2	1	2	1
3	3	1	1	3

Table 3.9

Then we can easily check that the axioms ($SSKU_2$) and ($KUSS_3$) are satisfied but the axiom ($SSKU_1$) is not satisfied.

Lemma 3.5. *$(X, \circ, 0)$ be a $SSKU$-algebra. Then the followings hold: for all $x, y, z \in X$,*

(1) $(x \circ (x \circ x)) \circ (x \circ (x \circ x)) = 0$,
(2) $x \circ 0 = 0 \circ 0$,
(3) *if* $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0 = (y \circ (z \circ z)) \circ (y \circ (z \circ z))$, *then*

$$(x \circ (z \circ z)) \circ (x \circ (z \circ z)) = 0,$$

(4) *if* $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$, *then*

$$[((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (z \circ (x \circ x))] \circ [((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (z \circ (x \circ x))] = 0,$$

(5) *if* $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$, *then*

$$[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (y \circ (z \circ z))] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (y \circ (z \circ z))] = 0,$$

(6) $[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ x)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ x)] = 0$,
(7) $[((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ (x \circ x)] \circ [((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ (x \circ x)] = 0$,
(8) *if* $(x \circ x) = x \circ (y \circ y)$, *then* $x \circ (x \circ (y \circ y)) = 0 \circ 0$,
(9) $(x \circ (x \circ x)) \circ (x \circ x) = x$,
(10) $(0 \circ 0) \circ (x \circ x) = x$,
(11) $(x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0)) = x$,
(12) $(0 \circ (x \circ x)) \circ (0 \circ (x \circ x)) = 0$,
(13) $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$.
(14) $((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (x \circ x) = ((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (y \circ y)$.

Proof. (1) In ($SSKU_1$), let $x = y = 0$ and $z = x$. Then we have

$0 =$
$[((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (((x \circ (0 \circ 0)) \circ (0 \circ (0 \circ 0))) \circ ((x \circ (0 \circ 0)) \circ (0 \circ (0 \circ 0))))]$
$\circ [((x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))) \circ (((x \circ (0 \circ 0)) \circ (0 \circ (0 \circ 0))) \circ ((x \circ (0 \circ 0)) \circ (0 \circ (0 \circ 0))))]$
$= [((x \circ x) \circ (x \circ x)) \circ (((x \circ x) \circ (0 \circ 0)) \circ ((x \circ x) \circ (0 \circ 0)))]$
$\circ [((x \circ x) \circ (x \circ x)) \circ (((x \circ x) \circ (0 \circ 0)) \circ ((x \circ x) \circ (0 \circ 0)))]$ [By ($SSKU_2$)]
$= [((x \circ x) \circ (x \circ x)) \circ (((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ (x \circ x)))]$
$\circ [((x \circ x) \circ (x \circ x)) \circ (((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ (x \circ x)))]$ [By ($SSKU_2$)]
$= [x \circ ((x \circ x)] \circ [x \circ ((x \circ x)]$ [By (S_2)]

(2) Let $x \in X$. Then we have

$x \circ 0 = ((x \circ x) \circ (x \circ x)) \circ ((0 \circ 0) \circ (0 \circ 0))$ [By (S_2)]

$= ((x \circ x) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ (0 \circ 0))$ [By (SSKU$_2$)]
$= ((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ (x \circ x))$ [By (S_1)]
$= 0 \circ 0.$ [By (S_2)]

(3) Suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0 = (y \circ (z \circ z)) \circ (y \circ (z \circ z))$ for all $x, y, z \in X$. Then we have

$0 =$
$[((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ(((x\circ(y\circ y))\circ(y\circ(z\circ z)))\circ((x\circ(y\circ y))\circ(y\circ(z\circ z))))]$
$\circ[((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ(((x\circ(y\circ y))\circ(y\circ(z\circ z)))\circ((x\circ(y\circ y))\circ(y\circ(z\circ z))))]$
[By exchanging x and z in (SSKU$_1$)]
$= [((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\circ((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (y \circ (x \circ x)))$
$\circ(((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (y \circ (x \circ x))))]$
$\circ[((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\circ((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (y \circ (x \circ x)))$
$\circ(((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (y \circ (x \circ x))))]$
[By (S_2)]
$= [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ (0 \circ 0)))]$
$\circ[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ (0 \circ 0)))]$
[By the hypothesis]
$= [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (0 \circ 0)] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (0 \circ 0)]$
[By (S$_2$)]
$= [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))]$
$\circ[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))]$
[By (SSKU$_2$)]
$= [x \circ (z \circ z)] \circ [x \circ (z \circ z)]$ [By (S_2)]

(4) Suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$ for all $x, y \in X$ and let $z \in X$. Then we have

$0 =$
$[((z\circ(y\circ y))\circ(z\circ(y\circ y)))\circ(((z\circ(x\circ x))\circ(x\circ(y\circ y)))\circ((z\circ(x\circ x))\circ(x\circ(y\circ y))))]$
$\circ[((z\circ(y\circ y))\circ(z\circ(y\circ y)))\circ(((z\circ(x\circ x))\circ(x\circ(y\circ y)))\circ((z\circ(x\circ x))\circ(x\circ(y\circ y))))]$
[By exchanging x and y in (SSKU$_1$)]
$= [((z \circ (y \circ y)) \circ (z \circ (y \circ y)))$
$\circ(((z \circ (x \circ x)) \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))))$
$\circ((z \circ (x \circ x)) \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))))))]$
$\circ[((z \circ (y \circ y)) \circ (z \circ (y \circ y)))$
$\circ(((z \circ (x \circ x)) \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))))$
$\circ((z \circ (x \circ x)) \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))))))]$
[By (S$_2$)]
$= [((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (((z \circ (x \circ x))) \circ (0 \circ 0)) \circ ((z \circ (x \circ x))) \circ (0 \circ 0)))]$
$\circ[((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (((z \circ (x \circ x))) \circ (0 \circ 0)) \circ ((z \circ (x \circ x))) \circ (0 \circ 0)))]$
[By the hypothesis]
$= [((z\circ(y\circ y))\circ(z\circ(y\circ y)))\circ((((z\circ(x\circ x))\circ((z\circ(x\circ x)))\circ(((z\circ(x\circ x))\circ((z\circ(x\circ x))))]$
$\circ[((z\circ(y\circ y))\circ(z\circ(y\circ y)))\circ((((z\circ(x\circ x))\circ((z\circ(x\circ x)))\circ(((z\circ(x\circ x))\circ((z\circ(x\circ x))))]$
[By (SSKU$_2$)]
$= [((z\circ(y\circ y))\circ(z\circ(y\circ y)))\circ(z\circ(x\circ x))]\circ[((z\circ(y\circ y))\circ(z\circ(y\circ y)))\circ(z\circ(x\circ x))]$
[By (S_2)]

(5) Suppose $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$ for all $x, y \in X$ and let $z \in X$. Then we have

$0 =$
$[((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ(((x\circ(y\circ y))\circ(y\circ(z\circ z)))\circ((x\circ(y\circ y))\circ(y\circ(z\circ z))))]$
$\circ[((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ(((x\circ(y\circ y))\circ(y\circ(z\circ z)))\circ((x\circ(y\circ y))\circ(y\circ(z\circ z))))]$
[By exchanging x and z in (SSKU$_1$)]
$= [((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\circ(((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \circ (y \circ (z \circ z)))$
$\circ((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \circ (y \circ (z \circ z))))]$
$\circ[((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$
$\circ(((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \circ (y \circ (z \circ z)))$
$\circ((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \circ (y \circ (z \circ z))))]$
[By (S_2)]
$= [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (((0 \circ 0) \circ (y \circ (z \circ z))) \circ ((0 \circ 0) \circ (y \circ (z \circ z)))]$
$\circ[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (((0 \circ 0) \circ (y \circ (z \circ z))) \circ ((0 \circ 0) \circ (y \circ (z \circ z)))]$
[By the hypothesis]
$= [((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ((((y\circ(z\circ z))\circ((y\circ(z\circ z)))\circ(((y\circ(z\circ z))\circ((y\circ(z\circ z))))]$
$\circ[((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ((((y\circ(z\circ z))\circ((y\circ(z\circ z)))\circ(((y\circ(z\circ z))\circ((y\circ(z\circ z))))]$
[By (S_1) and (SSKU$_2$)]
$= [((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ(y\circ(z\circ z))]\circ[((x\circ(z\circ z))\circ(x\circ(z\circ z)))\circ(y\circ(z\circ z))].$
[By (S_2)]

(6) Let $x, y \in X$. Then we have
$[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ x)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ x)]$
$= [(((y \circ y) \circ x) \circ ((y \circ y) \circ x)) \circ (x \circ x)] \circ [(((y \circ y) \circ x) \circ ((y \circ y) \circ x)) \circ (x \circ x)]$
[By (S_1)]
$= [(y \circ y) \circ (((x \circ (x \circ x)) \circ (x \circ (x \circ x)))] \circ [(y \circ y) \circ (((x \circ (x \circ x)) \circ (x \circ (x \circ x)))]$
[By (S_3)]
$= [(y \circ y) \circ 0] \circ [(y \circ y) \circ 0]$ [By (1)]
$= [0 \circ (y \circ y)] \circ [0 \circ (y \circ y)]$ [By (S_1)]
$= [0 \circ (y \circ (0 \circ 0))] \circ [0 \circ (y \circ (0 \circ 0))]$ [By (SSKU$_2$)]
$= [((0 \circ 0) \circ (0 \circ 0)) \circ (y \circ (0 \circ 0)))] \circ [((0 \circ 0) \circ (0 \circ 0)) \circ (y \circ (0 \circ 0)))]$
[By (S_2)]
$= [((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ y)] \circ [((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ y)]$
[By (S_1)]
$= (0 \circ 0) \circ (0 \circ 0)$ [By (S_2)]
$= 0.$ [By (S_2)]

(7) Let $x, y \in X$. Then we have
$[((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ (x \circ x)] \circ [((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ (x \circ x)]$
$= [0 \circ (x \circ x)] \circ [0 \circ (x \circ x)]$ [By (1)]
$= [0 \circ (x \circ (0 \circ 0))] \circ [0 \circ (x \circ (0 \circ 0))]$ [By (KUSS_2)]
$= [((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ x)] \circ [((0 \circ 0) \circ (0 \circ 0)) \circ ((0 \circ 0) \circ x)]$
[By (S_1) and (S_2)]
$= 0.$ [By (S_2)]

(8) Suppose $x \circ x = x \circ (y \circ y)$ for all $x, y \in X$. Then we have
$x \circ (x \circ (y \circ y))$
$= x \circ (x \circ x)$ [By the hypothesis]

$= ((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))$ [By (S_2)]
$= 0 \circ 0$. [By (1)]

(9) Let $x \in X$. Then we have
$(x \circ (x \circ x)) \circ (x \circ x) = (x \circ x) \circ (x \circ (x \circ x))$ [By (S_1)]
$= x$. [By (S_2)]

(10) Let $x \in X$. Then we get
$(0 \circ 0) \circ (x \circ x)$
$= (((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))) \circ (x \circ x)$ [By (1)]
$= (x \circ (x \circ x)) \circ (x \circ x)$ [By (S_2)]
$= x$. [By (9)]

(11) Let $x \in X$. Then we have
$(x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0))$
$= ((0 \circ 0) \circ x) \circ ((0 \circ 0) \circ x)$ [By (S_1)]
$= ((0 \circ 0) \circ ((x \circ x) \circ (x \circ x))) \circ ((0 \circ 0) \circ ((x \circ x) \circ (x \circ x)))$ [By (S_2)]
$= (x \circ x) \circ (x \circ x)$ [By (9)]
$= x$. [By (S_2)]

(12) Let $x \in X$. Then we get
$(0 \circ (x \circ x)) \circ (0 \circ (x \circ x)) = ((x \circ x) \circ 0) \circ ((x \circ x) \circ 0)$ [By (S_1)]
$= (0 \circ 0) \circ (0 \circ 0)$ [By (2)]
$= 0$. [By (S_2)]

(13) Let $x, y, z \in X$. Then we have
$x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = ((x \circ y) \circ (x \circ y)) \circ (z \circ z)$ [By (S_3)]
$= ((y \circ x) \circ (y \circ x)) \circ (z \circ z)$ [By (S_1)]
$= y \circ (x \circ ((z \circ z)) \circ (x \circ (z \circ z)))$. [By ($S_3$)]

(14) Let $a = (((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (x \circ x))$, $b = ((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (y \circ y)$ for all $x, y, z \in X$. Then from (S_1), (S_3) and (1), we can prove that

$$(a \circ (b \circ b)) \circ (a \circ (b \circ b)) = 0 = (b \circ (a \circ a)) \circ (b \circ (a \circ a)).$$

Thus by (SSKU_3), $a = b$. So the result holds. □

Lemma 3.6. *Let $(X, \circ, 0)$ be an SSKU-alebra. We define a binary relation $\leq$ on as follows: for all x, $y \in X$,*

$$x \leq y \text{ if and only if } x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0.$$

Then $\leq$ is an partial order on X.

Proof. The proof follows from Proposition 3.5(1), (SSKU_2) and Proposition 3.5(3). □

Proposition 3.7. *Let $(X, \circ, 0)$ be an SSKU-alebra. Then the following hold: for all $x, y, z \in X$,*

(1) *$x \leq y$ implies $z \circ (y \circ y) \leq z \circ (x \circ x)$, $x \circ (z \circ z) \leq y \circ (z \circ z)$,*
(2) *$x \leq y$ if and only if $y \circ y \leq x \circ x$,*
(3) *$y \leq x \circ (y \circ y)$,*
(4) *$(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq y \circ y$,*
(5) *$x \leq y$ implies $(x \circ (z \circ z)) \circ (x \circ (z \circ z)) \leq y$,*
(6) *$z \circ (x \circ (y \circ y)) \leq z \circ y$,*
(7) *$((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (x \circ x) \geq y \circ y$,*

(8) $z \circ (y \circ y) \leq z \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))$,
(9) $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq x$,
(10) $x \leq (x \circ (y \circ y)) \circ (y \circ y)$,
(11) $y \leq ((y \circ x) \circ (y \circ x)) \circ (x \circ x)$,
(12) $x \circ x = x \circ (y \circ y)$ *implies* $x \circ (x \circ (y \circ y)) = 0 \circ 0$,
(13) $0 \leq x$.

Proof. (1) The proof of the first part follows from (S_1) and Lemma 3.5(4). The proof of the second part follows from (S_1) and Lemma 3.5(5).

(2) Suppose $x \leq y$ for all $x, y \in X$. Then $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$. Thus we have

$$\begin{aligned}
&[(y \circ y) \circ (x \circ x) \circ (x \circ x))] \circ [(y \circ y) \circ (x \circ x) \circ (x \circ x))] \\
&= [(y \circ y) \circ x] \circ [(y \circ y) \circ x] \text{ [By } (S_2)] \\
&= (x \circ (y \circ y)) \circ (x \circ (y \circ y)) \text{ [By } (S_1)] \\
&= 0.
\end{aligned}$$

So by the definition of $\leq$, $y \circ y \leq x \circ x$. The proof of the converse is similar.

(3) Let $x, y \in X$. Then we have

$$\begin{aligned}
&[y \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \circ [y \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \\
&= [y \circ (((y \circ y) \circ x) \circ ((y \circ y) \circ x))] \circ [y \circ (((y \circ y) \circ x) \circ ((y \circ y) \circ x))] \text{ [By } (S_1)] \\
&= [((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ x] \circ [((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ x] \text{ [By } (S_3)] \\
&= (0 \circ x) \circ (0 \circ x) \text{ [By Lemma 3.5(1)]} \\
&= (0 \circ 0) \circ (0 \circ 0) \text{ [By (} S_1 \text{ and Lemma 3.5(2)]} \\
&= 0. \text{ [By } (S_2)]
\end{aligned}$$

Thus $y \leq x \circ (y \circ y)$.

(4) The proof follows from (3) and (2).

(5) Suppose $x \leq y$ for all $x, y \in X$. Then $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$. Let $z \in X$. Then we have

$$\begin{aligned}
&[((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (y \circ y)] \circ [((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \circ (y \circ y)] \\
&= [(((z \circ z) \circ x) \circ ((z \circ z) \circ x)) \circ (y \circ y)] \circ [(((z \circ z) \circ x) \circ ((z \circ z) \circ x)) \circ (y \circ y)] \\
&\quad \text{[By } (S_1)] \\
&= [(z \circ z) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \circ [(z \circ z) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \\
&\quad \text{[By } (S_2)] \\
&= [(z \circ z) \circ 0] \circ [(z \circ z) \circ 0] \\
&= (0 \circ 0) \circ (0 \circ 0) \text{ [By Lemma 3.5(2)]} \\
&= 0. \text{ [By } (S_2)]
\end{aligned}$$

Thus $(x \circ (z \circ z)) \circ (x \circ (z \circ z)) \leq y$.

(6) Let $x, y, z \in X$. Then by (4), $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq y \circ y$. Thus by the first part of Lemma 3.7(1), we have

$$z \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))) \leq z \circ ((y \circ y) \circ (y \circ y)).$$

So by (S_2), $z \circ (x \circ (y \circ y)) \leq z \circ y$.

(7) Let $x, y, z \in X$. Then we have

$$\begin{aligned}
&((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (x \circ x) \\
&= (x \circ x) \circ (((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \text{ [By } (S_1)] \\
&\geq (x \circ x) \circ ((x \circ ((y \circ y) \circ (y \circ y))) \circ (x \circ ((y \circ y) \circ (y \circ y)))) \text{ [By (6)]} \\
&= (x \circ x) \circ ((x \circ y) \circ (x \circ y)) \text{ [By } (S_2)] \\
&= (((x \circ x) \circ x) \circ ((x \circ x) \circ x)) \circ y \text{ [By } (S_3)]
\end{aligned}$$

$= (0 \circ 0) \circ y$ [By Lemma 3.5(1)]
$= y \circ y$. [By (S_1) and ($SSKU_2$)]
Thus $((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \circ (x \circ x) \geq y \circ y$.

(8) The proof follows from (3) and the first part of (1).

(9) Let $x, y \in X$. Then we have

$$\begin{aligned}
&((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ x)] \circ [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ x)] \\
&= [(x \circ x) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \circ [(x \circ x) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \\
&\quad \text{[By } (S_1)] \\
&= [((x \circ x) \circ x) \circ (x \circ x) \circ x)) \circ (y \circ y)] \circ [((x \circ x) \circ x) \circ (x \circ x) \circ x)) \circ (y \circ y)] \text{ [By } (S_3)] \\
&= (0 \circ (y \circ y)) \circ (0 \circ (y \circ y)) \text{ [By } (S_1) \text{ and Lemma 2.6(1)]} \\
&= (0 \circ 0) \circ (0 \circ 0) \text{ [By } (S_1) \text{ and Lemma 2.6(1)]} \\
&= 0. \text{ [By } (S_1)]
\end{aligned}$$

Thus $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \leq x$.

(10) Let $x, y \in X$. Then we get

$$\begin{aligned}
&[x \circ ((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))] \\
&\circ [x \circ ((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))] \\
&= [(x \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \\
&\quad \circ [(x \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y)))] \text{ [By } (S_1) \text{ and } (S_3)] \\
&= 0 \text{ [By Lemma 2.6(1)]}
\end{aligned}$$

Thus $x \leq (x \circ (y \circ y)) \circ (y \circ y)$.

(11) Let $x, y \in X$. Then we have

$$\begin{aligned}
&[y \circ (((y \circ x) \circ (y \circ x)) \circ (x \circ x))] \circ [y \circ (((y \circ x) \circ (y \circ x)) \circ (x \circ x))] \\
&= [y \circ (((y \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))) \circ ((y \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))))] \\
&\quad \circ [y \circ (((y \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))) \circ ((y \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))))] \\
&\quad \text{[By } (S_3)] \\
&= [y \circ ((y \circ 0) \circ (y \circ 0))] \circ [y \circ ((y \circ 0) \circ (y \circ 0))] \text{ [By Lemma 2.6(1)]} \\
&= [y \circ ((0 \circ 0) \circ (0 \circ 0))] \circ [y \circ ((0 \circ 0) \circ (0 \circ 0))] \text{ [By Lemma 2.6(2)]} \\
&= (y \circ 0) \circ (y \circ 0) \text{ [By } (S_2)] \\
&= 0. \text{ [By Lemma 2.6(2) and (12)]}
\end{aligned}$$

Thus the inequality holds.

(12) Suppose $x \circ x = x \circ (y \circ y)$ for all $x, y \in X$. Then we have

$$\begin{aligned}
&x \circ (x \circ (y \circ y)) \\
&= x \circ (x \circ x) \\
&= ((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \text{[By } (S_2)] \\
&= 0 \circ 0. \text{ [By Lemma 2.6(1)]}
\end{aligned}$$

(13) The proof follows from Lemma 3.5(2) and (S_2) □

Proposition 3.8. *Let $(X, \circ, 0)$ be an SSKU-algebra. We define a binary operation $*$ on X as follows: for all $x, y \in X$,*

$$x * y = (y \circ (x \circ x)) \circ (y \circ (x \circ x)).$$

Then $(X, *, 0)$ is a KU-algebra.
In this case, $(X, *, 0)$ is called a *KU-algebra induced by* $(X.\circ, 0)$.

Proof. Let $x, y, z \in X$. Then by (S_2), (S_3) and ($SSKU_1$), we have

$$\begin{aligned}
&(x * y) * [(y * z) * (x * z)] \\
&= [((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (((z \circ (y \circ y)) \circ (y \circ (x \circ x))) \circ ((z \circ (y \circ y)) \circ (y \circ (x \circ x))))]
\end{aligned}$$

$\circ[((z\circ(x\circ x))\circ(z\circ(x\circ x)))\circ(((z\circ(y\circ y))\circ(y\circ(x\circ x)))\circ((z\circ(y\circ y))\circ(y\circ(x\circ x))))]$
$= 0.$

Thus the condition (KU_1) holds.

Let $x \in X$. Then we get

$$\begin{aligned} x * 0 &= (0 \circ (x \circ x)) \circ (0 \circ (x \circ x)) \\ &= (0 \circ 0) \circ (0 \circ 0) \text{ [By } (SSKU_1)] \\ &= 0, \text{ [By } (S_2)] \\ 0 * x &= (x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0)) \\ &= (x \circ x) \circ (x \circ x) \text{ [By } (SSKU_2)] \\ &= x. \text{ [By } (S_2)] \end{aligned}$$

Thus the conditions (KU_2) and (KU_3) hold.

Finally, suppose $x * y = 0 = y * x$ for all $x, y \in X$. Then we have

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)) = 0 = (x \circ (y \circ y)) \circ (x \circ (y \circ y)).$$

Thus by ($SSKU_3$), $x = y$. So the condition (KU_4). Hence $(X, *, 0)$ is a KU-algebra. □

Example 3.9. (1) Consider the $SSKU$-algebra $(X, \circ, 0) = (\{0, x, y, 1\}, \circ, 0)$ given in Example 3.2(1). Then from $(X, \circ, 0)$, we obtain the KU-algebra $(X, *, 0)$ having the following Cayley table:

$*$	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

Table 3.10

(2) Let $(X, \circ, 0)$ be the $SSKU$-algebra given in Example 3.2(3). Then we have the KU-algebra $(X, *, 0)$ having the following Cayley table:

$*$	0	1	2	3	4	5
0	0	1	2	3	3	3
1	0	0	2	2	2	1
2	0	1	0	1	0	1
3	0	0	0	0	0	0
4	0	0	2	3	4	4
5	0	0	0	0	0	0

Table 3.11

Definition 3.10. Let $(X, \circ, 0)$ be a $SSKU$-algebra. Then X is said to be:

(i) *commutative*, if for all $x, y \in X$,

$$x \dot{\wedge} y = y \dot{\wedge} x, \tag{3.1}$$

where $x \dot{\wedge} y = [(x \circ (x \circ (y \circ y)))] \circ [(x \circ (x \circ (y \circ y)))]$,

(ii) *bounded*, if there is the greatest element 1 of X (called the unit of X), i.e.,

$$(x \circ (1 \circ 1)) \circ (x \circ (1 \circ 1)) = 0 \text{ for each } x \in X$$

and $(1 \circ (x \circ x)) \circ (1 \circ (x \circ x))$ will be denoted by Nx.

Example 3.11. (1) Consider the Cayley table for the binary operation $\circ$ on the set $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	0	3	1	2
1	3	1	3	1
2	1	3	2	3
3	2	1	3	0

Table 3.12

Then we can see that $(X, \circ, 0)$ is a commutative $SSKU$-algebra.

(2) Consider the Cayley table for the binary operation $\circ$ on the set $X = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	0	3	3	3
1	3	1	1	1
2	3	1	2	2
3	3	1	2	3

Table 3.13

Then $(X, \circ, 0)$ is a bounded $SSKU$-algebra.

Lemma 3.12. *Let $(X, \circ, 0)$ be a $SSKU$-algebra. If X is commutative, then $x \dot{\wedge} y \leq x$, $x \dot{\wedge} y \leq y$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. Then we have

$[(x \dot{\wedge} y) \circ (x \circ x)] \circ [(x \dot{\wedge} y) \circ (x \circ x)]$
$= [((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (x \circ x)] \circ [((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (x \circ x)]$
$= [((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (y \circ y))] \circ [((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \circ (x \circ (y \circ y))]$
[By (S_1) and (S_3)]
$= [0 \circ (x \circ (y \circ y))] \circ [0 \circ (x \circ (y \circ y))]$ [By Lemma 3.5(1)]
$= 0$, [By Lemma 3.5(2) and (S_2)]

$[(x \dot{\wedge} y) \circ (y \circ y)] \circ [(x \dot{\wedge} y) \circ (y \circ y)]$
$= [(y \dot{\wedge} x) \circ (y \circ y)] \circ [(y \dot{\wedge} x) \circ (y \circ y)]$ [Since X is commutative]
$= [((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x)))) \circ (y \circ y)] \circ [((y \circ (y \circ (x \circ x))) \circ (y \circ (y \circ (x \circ x)))) \circ (y \circ y)]$
$= [((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ (y \circ (x \circ x))] \circ [((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \circ (y \circ (x \circ x))]$
[By (S_1) and (S_3)]
$= [0 \circ (y \circ (x \circ x))] \circ [0 \circ (y \circ (x \circ x))]$ [By Lemma 3.5(1)]
$= 0$. [By Lemma 3.5(2) and (S_2)]

Thus $x \dot{\wedge} y \leq x$, $x \dot{\wedge} y \leq y$. $\square$

Proposition 3.13. *Let $(X, \circ, 0)$ be a bounded SSKU-algebra. Then the following hold: for all $x, y \in X$,*
(1) $N1 = 0,\ N0 = 1$,
(2) $Nx \circ (y \circ y) = Ny \circ (x \circ x)$,
(3) $NNx \leq x$.

Proof. (1) Let $x \in X$. Then we have,

$$\begin{aligned} N0 &= (1 \circ (0 \circ 0)) \circ (1 \circ (0 \circ 0)) \\ &= (1 \circ 1) \circ (1 \circ 1) \text{ [By (SSKU}_2)] \\ &= 1, \text{ [By (S}_2)] \\ N1 &= (1 \circ (1 \circ 1)) \circ (1 \circ (1 \circ 1)) \\ &= 0. \text{ [By Lemma 3.5(1)]} \end{aligned}$$

(2) Let $x, y \in X$. Then we have

$$\begin{aligned} Nx \circ (y \circ y) &= ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))) \circ (y \circ y) \\ &= (((x \circ x) \circ 1) \circ ((x \circ x) \circ 1)) \circ (y \circ y) \text{ [By (S}_1)] \\ &= (x \circ x) \circ ((1 \circ (y \circ y)) \circ (1 \circ (y \circ y))) \text{ [By (S}_3)] \\ &= (x \circ x) \circ Ny \\ &= Ny \circ (x \circ x). \text{ [By (S}_1)] \end{aligned}$$

(3) Let $x \in X$. Then we have

$$\begin{aligned} & NNx \circ (x \circ x) \\ &= [(1 \circ (Nx \circ Nx)) \circ (x \circ x)] \circ [(1 \circ (Nx \circ Nx)) \circ (x \circ x)] \\ &= [(1 \circ (((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))) \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x)))) \circ (x \circ x)] \\ &\quad \circ [(1 \circ (((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))) \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x)))) \circ (x \circ x)] \\ &= [((1 \circ (1 \circ (x \circ x))) \circ (1 \circ (1 \circ (x \circ x)))) \circ (x \circ x)] \\ &\quad \circ [((1 \circ (1 \circ (x \circ x))) \circ (1 \circ (1 \circ (x \circ x)))) \circ (x \circ x)] \text{ [By (S}_2)] \\ &= [(1 \circ (x \circ x)) \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x)))] \\ &\quad \circ [(1 \circ (x \circ x)) \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x)))] \text{ [By (S}_1) \text{ and (S}_3)] \\ &= ((1 \circ (x \circ x)) \circ 0) \circ ((1 \circ (x \circ x)) \circ 0) \text{ [By Lemma 3.5(1)]} \\ &= (0 \circ 0) \circ (0 \circ 0) \text{ [By Lemma 3.5(2)]} \\ &= 0. \text{ [By Lemma 3.5(1)]} \end{aligned}$$

Thus $NNx \leq x$. □

Proposition 3.14. *Let $(X, \circ, 0)$ be a bounded commutative SSKU-algebra. Then the following hold: for all $x, y \in X$,*
(1) $NNx = x$,
(2) $x \leq y$ *implies* $Ny \leq Nx$,
(3) $x \dot{\wedge} 1 = x,\ 1 \dot{\wedge} x = 1$,
(4) $Nx \dot{\wedge} Ny = N(y \dot{\vee} x),\ Nx \dot{\vee} Ny = N(x \dot{\wedge} y)$, *where* $x \dot{\vee} y = N(Nx \dot{\wedge} Ny)$.

Proof. (1) Let $x \in X$. Then we get

$$\begin{aligned} & [x \circ (NNx \circ NNx)] \circ [x \circ (NNx \circ NNx)] \\ &= [x \circ (((1 \circ (Nx \circ Nx)) \circ (1 \circ (Nx \circ Nx))) \circ ((1 \circ (Nx \circ Nx)) \circ (1 \circ (Nx \circ Nx))))] \\ &\quad \circ [x \circ (((1 \circ (Nx \circ Nx)) \circ (1 \circ (Nx \circ Nx))) \circ ((1 \circ (Nx \circ Nx)) \circ (1 \circ (Nx \circ Nx))))] \\ &= [x \circ (1 \circ (Nx \circ Nx))] \circ [x \circ (1 \circ (Nx \circ Nx))] \text{ [By (S}_2)] \\ &= [x \circ (1 \circ (((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))) \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x)))] \\ &\quad \circ [x \circ (1 \circ (((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))) \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x)))] \\ &= [x \circ (1 \circ (1 \circ (x \circ x)))] \circ [x \circ (1 \circ (1 \circ (x \circ x)))] \text{ [By (S}_2)] \\ &= [x \circ (x \circ (x \circ (1 \circ 1)))] \circ [x \circ (x \circ (x \circ (1 \circ 1)))] \text{ [Since } X \text{ is commutative]} \end{aligned}$$

$= 0$. [Since 1 is the unit of X]

Thus $x \leq NNx$. By Proposition 3.12(3), $NNx \leq x$. So $NNx = x$.

(2) Suppose $x \leq y$ for all $x, y \in X$. Then we have

$(Ny \circ (Nx \circ Nx)) \circ (Ny \circ (Nx \circ Nx))$

$= [((1 \circ (y \circ y)) \circ (1 \circ (y \circ y))) \circ (1 \circ (x \circ x))]$
$\circ[((1 \circ (y \circ y)) \circ (1 \circ (y \circ y))) \circ (1 \circ (x \circ x))]$ [By (S$_2$)]

$= [((\circ 1 \circ ((x \circ x) \circ 1)) \circ (\circ 1 \circ ((x \circ x) \circ 1))) \circ (y \circ y)]$
$\circ[((\circ 1 \circ ((x \circ x) \circ 1)) \circ (\circ 1 \circ ((x \circ x) \circ 1))) \circ (y \circ y)]$ [By (S$_1$) and (S$_3$)]

$= [((1 \circ (1 \circ (x \circ x))) \circ (1 \circ (1 \circ (x \circ x)))) \circ (y \circ y)]$
$\circ[((1 \circ (1 \circ (x \circ x))) \circ (1 \circ (1 \circ (x \circ x)))) \circ (y \circ y)]$ [By (S$_1$)]

$= ((x \circ (x \circ (1 \circ 1))) \circ (x \circ (x \circ (1 \circ 1)))) \circ (y \circ y)]$
$\circ((x \circ (x \circ (1 \circ 1))) \circ (x \circ (x \circ (1 \circ 1)))) \circ (y \circ y)]$ [Since X is commutative]

$= [((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ (1 \circ 1))]$
$\circ[((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ (x \circ (1 \circ 1))]$ [By (S$_1$) and (S$_3$)]

$= (0 \circ (x \circ (1 \circ 1))) \circ (0 \circ (x \circ (1 \circ 1)))$ [By the hypothesis]

$= (0 \circ 0) \circ (0 \circ 0)$ [By Lemma 3.5(2)]

$= 0$. [By (S$_2$)]

Thus $Ny \leq Nx$.

(3) Let $x \in X$. Then we have

$[(x\dot{\wedge}1) \circ (x \circ x)] \circ [(x\dot{\wedge}1) \circ (x \circ x)]$

$= [((x\circ(x\circ(1\circ1)))\circ(x\circ(x\circ(1\circ1))))\circ(x\circ x)]\circ[((x\circ(x\circ(1\circ1)))\circ(x\circ(x\circ(1\circ1))))\circ(x\circ x)]$

$= [((x\circ(x\circ x))\circ(x\circ(x\circ x)))\circ(x\circ(1\circ1))]\circ[((x\circ(x\circ x))\circ(x\circ(x\circ x)))\circ(x\circ(1\circ1))]$
[By (S$_1$) and (S$_3$)]

$= [0 \circ (x \circ (1 \circ 1))] \circ [0 \circ (x \circ (1 \circ 1))]$ [By Lemma 3.5(1)]

$= (0 \circ 0) \circ (0 \circ 0)$ [By Lemma 3.5(2)]

$= 0$, [By (S$_2$)]

$[x \circ ((x\dot{\wedge}1) \circ (x\dot{\wedge}1))] \circ [x \circ ((x\dot{\wedge}1) \circ (x\dot{\wedge}1))]$

$= [x\circ(((x\circ(x\circ(1\circ1)))\circ(x\circ(x\circ(1\circ1))))\circ((x\circ(x\circ(1\circ1)))\circ(x\circ(x\circ(1\circ1)))))]$
$\circ[x\circ(((x\circ(x\circ(1\circ1)))\circ(x\circ(x\circ(1\circ1))))\circ((x\circ(x\circ(1\circ1)))\circ(x\circ(x\circ(1\circ1)))))]$

$= [x \circ (x \circ (x \circ (1 \circ 1)))] \circ [x \circ (x \circ (x \circ (1 \circ 1)))]$ [By (S$_2$)]

$= 0$, [Since 1 is the unit of X]

$((1\dot{\wedge}x) \circ (1 \circ 1)) \circ ((1\dot{\wedge}x) \circ (1 \circ 1))$

$= [((1\circ(1\circ(x\circ x)))\circ(1\circ(1\circ(x\circ x))))\circ(1\circ1)]\circ[((1\circ(1\circ(x\circ x)))\circ(1\circ(1\circ(x\circ x))))\circ(1\circ1)]$

$= [((1\circ(1\circ1))\circ(1\circ(1\circ1)))\circ(1\circ(x\circ x))]\circ[((1\circ(1\circ1))\circ(1\circ(1\circ1)))\circ(1\circ(x\circ x))]$
[By (S$_1$) and (S$_3$)]

$= [0 \circ (1 \circ (x \circ x))] \circ [0 \circ (1 \circ (x \circ x))]$ [By Lemma 3.5(1)]

$= 0$, [By Lemma 3.5(2) and (S$_2$)]

$[1 \circ ((x\dot{\wedge}1) \circ (x\dot{\wedge}1))] \circ [1 \circ ((x\dot{\wedge}1) \circ (x\dot{\wedge}1))]$

$= [1 \circ ((1\dot{\wedge}x) \circ (1\dot{\wedge}x))] \circ [1 \circ ((1\dot{\wedge}x) \circ (1\dot{\wedge}x))]$ [Since X is commutative]

$= [1\circ((1\circ(1\circ(x\circ x)))\circ(1\circ(1\circ(x\circ x))))]\circ[1\circ((1\circ(1\circ(x\circ x)))\circ(1\circ(1\circ(x\circ x))))]$

$= [((1 \circ 1) \circ (1 \circ 1)) \circ (1 \circ (x \circ x))] \circ [((1 \circ 1) \circ (1 \circ 1)) \circ (1 \circ (x \circ x))]$ [By (S$_3$)]

$= (1 \circ (1 \circ (x \circ x))) \circ (1 \circ (1 \circ (x \circ x)))$ [By (S$_1$)]

$= (x \circ (x \circ (1 \circ 1))) \circ (x \circ (x \circ (1 \circ 1)))$ [Since X is commutative]

$= 0$. [Since X is the unit of X]
Thus by (SSKU$_3$), $x\dot{\wedge}1 = x,\ 1\dot{\wedge}x = 1$.
(4) Let $x, y \in X$. Then we have

$$\begin{aligned} Nx\dot{\wedge}Ny &= N(x\dot{\vee}y) \\ &= N(Nx\dot{\wedge}Ny) \\ &= N(Ny\dot{\wedge}Nx) \text{ [Since } X \text{ is commutative]} \\ &= N(y\dot{\vee}x). \end{aligned}$$

Similarly, we can show that the second part holds. □

Theorem 3.15. *Let $(X, \circ, 0)$ be a bounded SSKU-algebra. The the following are equivalent: for all $x, y \in X$,*
(1) $NNx = x$,
(2) $x \circ (y \circ y) = Ny \circ (Nx \circ Nx)$,
(3) $x \circ (Ny \circ Ny) = y \circ (Nx \circ Nx)$,
(4) $x \le Ny$ *implies* $y \le Nx$.

Proof. (1)⇒(2) Suppose $NNx = x$ for all $x \in X$ and let $y \in X$. Then we have

$$\begin{aligned} x \circ (y \circ y) &= NNx \circ (y \circ y) \text{ [By the hypothesis]} \\ &= Ny \circ (Nx \circ Nx) \text{ [By Proposition 3.11(2)]} \end{aligned}$$

(2)⇒(3) Suppose $x \circ (y \circ y) = Ny \circ (Nx \circ Nx)$ and let $x, y \in X$. Then we have

$$x \circ (Ny \circ Ny) = NNy \circ (Nx \circ Nx),\ \ y \circ (Nx \circ Nx) = NNx \circ (Ny \circ Ny).$$

Thus by Proposition 3.11(2), $NNy \circ (Nx \circ Nx) = NNx \circ (Ny \circ Ny)$. So we get

$$x \circ (Ny \circ Ny) = y \circ (Nx \circ Nx).$$

(3)⇒(4) Suppose (3) holds and $x \le Ny$ for all $x, y \in X$. Then clearly,

$$(x \circ (Ny \circ Ny)) \circ (x \circ (Ny \circ Ny)) = 0.$$

Thus by (3), $(y \circ (Nx \circ Nx)) \circ (y \circ (Nx \circ Nx)) = 0$. So $y \le Nx$.

(4)⇒(1) Suppose (4) holds and let $x \in X$. Then by Lemma 3.12(3), $NNx \le x$. If $x \le Ny$ for all $y \in X$, then clearly, $x \le Nx$. Thus by the hypothesis, $x \le Nx$. Also, by the hypothesis, $x \le NNx$. So $NNx = x$. □

Theorem 3.16. *Let $(X, \circ, 0)$ be a SSKU-algebra. If $x\dot{\wedge}y = 0$ for all $x, y \in X$, then the the following are equivalent: for all $x, y \in X$,*
(1) *X is commutative,*
(2) $x\dot{\wedge}y \le y\dot{\wedge}x$,
(3) $(x\dot{\wedge}y) \circ (y\dot{\wedge}x) = 0$.

Proof. The proofs follows from Lemma 3.5 and (S$_2$). □

A partially ordered set $(L, \le)$ is called a *lower* or *meet semilattice*, if every pair of elements in L has a greatest lower bound and it is called an *upper* or a *join semilattice*, if every pair of elements in L has a least upper bound. Furthermore, it is called a *lattice*, if it is both an upper and a lower semilattice. For all $x, y \in L$, the meet and the join of $\{x, y\}$ will be denoted by $x \wedge y = glb\{x, y\}$ and $x \vee y = lub\{x, y\}$ (See [33]).

Proposition 3.17. *Let $(X, \circ, 0)$ be a SSKU-algebra. If X is commutative, then $x\dot{\wedge}y$ is a greatest lower bound of $\{x, y\}$ for all $x, y \in X$, i.e., $(X \leq)$ is a meet semilattice.*

Proof. Suppose X is commutative and let $x, y \in X$. Then Lemma 3.12, $x\dot{\wedge}y \leq x$, $x\dot{\wedge}y \leq y$. Thus $x\dot{\wedge}y$ is a lower bound of $\{x, y\}$. Let $z \in X$ such that $z \leq x$, $z \leq y$. Then we get

$$(z \circ (x \circ x)) \circ (z \circ (x \circ x)) = 0 = (z \circ (y \circ y)) \circ (z \circ (y \circ y)). \tag{3.2}$$

Thus we have

$z = (z \circ (0 \circ 0)) \circ (z \circ (0 \circ 0))$ [By Lemma 3.5(11)]
$= [z \circ (((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ ((z \circ (x \circ x)) \circ (z \circ (x \circ x))))]$
$\circ[z \circ (((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ ((z \circ (x \circ x)) \circ (z \circ (x \circ x))))]$ [By (3.4)]
$= [z \circ (z \circ (x \circ x))] \circ [z \circ (z \circ (x \circ x))]$ [By (S_1)]
$= [x \circ (x \circ (z \circ z))] \circ [x \circ (x \circ (z \circ z))]$. [Since X is commutative]

Similarly, $z = y \circ (y \circ (z \circ z))] \circ [y \circ (y \circ (z \circ z))]$. Thus we get

$z = [x \circ (x \circ (z \circ z))] \circ [x \circ (x \circ (z \circ z))]$
$= [x \circ (x \circ (((y \circ (y \circ (z \circ z))) \circ (y \circ (y \circ (z \circ z)))) \circ ((y \circ (y \circ (z \circ z))) \circ (y \circ (y \circ (z \circ z))))))]$
$\circ[x \circ (x \circ (((y \circ (y \circ (z \circ z))) \circ (y \circ (y \circ (z \circ z)))) \circ ((y \circ (y \circ (z \circ z))) \circ (y \circ (y \circ (z \circ z))))))]$
$= [x \circ (x \circ (y \circ (y \circ (z \circ z))))] \circ [x \circ (x \circ (y \circ (y \circ (z \circ z))))]$ [By (S_2)]
$\leq (x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))$
$= x\dot{\wedge}y$.

So $x\dot{\wedge}y$ is the greatest lower bound of $\{x, y\}$. □

Proposition 3.18. *Let $(X, \circ, 0)$ be a SSKU-algebra. If X is bounded and commutative, then $x\dot{\vee}y$ is a least upper bound of $\{x, y\}$ for all $x, y \in X$, i.e., $(X \leq)$ is a join semilattice.*

Proof. Suppose X is bounded and commutative and let $x, y \in X$. Then clearly, $Nx\dot{\wedge}Ny \leq Nx$, $Nx\dot{\wedge}Ny \leq Ny$ by Lemma 3.12. By Proposition 3.14(1), we have

$$x = NNx \leq N(Nx\dot{\wedge}Ny) = x\dot{\vee}y,\ y = NNy \leq N(Nx\dot{\wedge}Ny) = x\dot{\vee}y.$$

Thus $x\dot{\vee}y$ is an upper bound of $\{x, y\}$. Let $z \in X$ such that $x \leq z$, $y \leq z$. Then by Proposition 3.14(2), $Nz \leq Nx$, $Nz \leq Ny$. Thus $Nz \leq Nx\dot{\wedge}Ny$. Also, by Proposition 3.14(2), $N(Nx\dot{\wedge}Ny) \leq NNz$. Thus $x\dot{\vee}y \leq z$. So $x\dot{\vee}y$ is a least upper bound of $\{x, y\}$. Hence X is a join semilattice. □

From Propositions 3.17 and 3.18, we have the following.

Corollary 3.19. *Let $(X, \circ, 0)$ be a SSKU-algebra. If X is bounded and commutative, then $(X \leq)$ is a lattice.*

4. Congruences on $SSKU$-algebras

In this section, we obtain some results on the images and the preimages of $SSKU$-ideals of a $SSKU$-algebra under a $SSKU$-homomorphism. Also, we will deal with congruences by ideals $kerf$ and $Kerf$ of a $SSKU$-homomorphism f.

Definition 4.1. Let $(X, \circ, 0)$ be a $SSKU$-algebra and I a nonempty subset of X. Then I is called a *Shefer stroke KU-ideal* (briefly, $SSKU$-deal) of X, if it satisfies the following conditions: for all $x, y \in X$,

($SSKUI_1$) $0 \in I$,
($SSKUI_2$) $(y \circ (x \circ x)) \circ (y \circ (x \circ x)) \in I$ and $x \in I$ imply $y \in I$.

It is clear that $\{0\}$ and X are two $SSKU$-ideals of X. We will call X a *trivial $SSKU$-ideal*. An $SSKU$-ideal I said to be *proper*, if $I \neq X$.

Example 4.2. Let $(X, \circ, 0)$ be the $SSKU$-algebra given in Example 3.2(2). Then the subsets $\{0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{0, 1, 2\}$, $\{0, 1, 3\}$ and $\{0, 2, 3\}$ of X all are proper $SSKU$-ideals of X.

Proposition 4.3. *Let $(X, \circ, 0)$ be a $SSKU$-algebra, I a $SSKU$-deal of X and $y \in I$. If $x \leq y$ for each $x \in X$, then $x \in I$.*

Proof. Suppose $x \leq y$ for each $x \in X$. Then clearly, $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) = 0$. Thus by ($SSKUI_1$), $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in I$. Since $y \in I$, by ($SSKUI_2$), $x \in X$. □

For a $SSKU$-algebra X and any $a, b \in X$, define the subset $A(a, b)$ of X as follows:

$$A(a, b) = \{x \in X : (x \circ (a \circ a)) \circ (x \circ (a \circ a)) \leq b\}.$$

Theorem 4.4. *Let $(X, \circ, 0)$ be a $SSKU$-algebra and I a nonempty subset of X. Then I is a $SSKU$-deal of X if and only if $A(x, y) \subset I$ for all $x, y \in I$.*

Proof. Suppose I is a $SSKU$-deal of X and let $z \in A(x, y)$ for all $x, y \in I$. Then $(z \circ (x \circ x)) \circ (z \circ (x \circ x)) \leq y$. Thus by Proposition 4.3, $(z \circ (x \circ x)) \circ (z \circ (x \circ x)) \in I$. So by ($SSKUI_2$), $z \in I$. Hence $A(x, y) \subset I$.

Conversely, suppose $A(x, y) \subset I$ for all $x, y \in I$. Since I is a nonempty subset of X, there $x \in I$. Then by Lemmas 3.5(2) and 3.6(13),

$$(0 \circ (x \circ x)) \circ (0 \circ (x \circ x)) = 0 \leq x.$$

Thus $0 \in A(x, x) \subset I$. So ($SSKUI_1$) holds.

Now suppose $(y \circ (x \circ x)) \circ (y \circ (x \circ x)) \in I$ and $x \in I$ for all $x, y \in X$. Then by Lemma 3.5(6) and Lemma 3.6(13),

$$[((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ (x \circ x)] \circ [((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ (x \circ x)] \leq x.$$

Thus $y \in A((y \circ (x \circ x)) \circ (y \circ (x \circ x)), x) \subset I$. So ($SSKUI_2$) holds. Hence I is an ideal of X. □

The following is an immediate consequence of Theorem 4.4.

Corollary 4.5. *Let $(X, \circ, 0)$ be a $SSKU$-algebra and I a nonempty subset of X. Then I is a $SSKU$-deal of X if and only if the following condition holds: for all $x, y \in I$ and each $z \in X$,*

$$[((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (y \circ y)] \circ [((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (y \circ y)] = 0 \text{ imply } z \in I.$$

Definition 4.6. Let $(X, \circ, 0)$ be an $SSKU$-algebra and R an equivalence relation on X. Then R is called a *Sheffer stroke KU-congruence* (briefly, $SSKU$-congruence) on X, if for all $x, y, u, v \in X$, xRy and uRv imply that

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)) R (v \circ (u \circ u)) \circ (v \circ (u \circ u)),$$

$$(x \circ (y \circ y)) \circ (x \circ (y \circ y)) R (u \circ (v \circ v)) \circ (u \circ (v \circ v)).$$

Proposition 4.7. *Let $(X, \circ, 0)$ be an associative SSKU-algebra and I a SSKU-deal of X. We define the relation $\backsim_I$ on X as follows: for all $x, y \in X$,*

$$x \sim_I y \text{ iff } (y \circ (x \circ x)) \circ (y \circ (x \circ x)),\ (x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in I.$$

Then $\backsim_I$ is an SSKU-congruence on X.

Proof. It is obvious that $\backsim_I$ is symmetric Let $x \in X$. Then clearly, $(x \circ (x \circ x)) \circ (x \circ (x \circ x)) = 0$ by Lemma 3.5(1). By (SSKUI$_1$), $(x \circ (x \circ x)) \circ (x \circ (x \circ x)) \in I$. Thus $x \sim_I x$. So $\sim_I$ is reflexive. Suppose $x \sim_I y$ and $y \sim_I z$ for all $x, y, z \in X$. Then clearly,

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)),\ (x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in I, \tag{4.1}$$

$$(y \circ (z \circ z)) \circ (y \circ (z \circ z)),\ (z \circ (y \circ y)) \circ (z \circ (y \circ y)) \in I. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned}
&[(((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (z \circ (y \circ y))) \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x)))] \\
&\circ[(((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (z \circ (y \circ y))) \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x)))] \\
&= [(((z \circ (x \circ x)) \circ (0 \circ 0)) \circ (z \circ (y \circ y))) \circ ((y \circ (x \circ x)) \circ (0 \circ 0))] \\
&\circ[(((z \circ (x \circ x)) \circ (0 \circ 0)) \circ (z \circ (y \circ y))) \circ ((y \circ (x \circ x)) \circ (0 \circ 0)) \text{ [By (SSKU}_2)] \\
&= [(((z \circ (x \circ x)) \circ (z \circ (y \circ y))) \circ (y \circ (x \circ x))) \circ ((0 \circ 0) \circ (0 \circ 0))] \\
&\circ[(((z \circ (x \circ x)) \circ (z \circ (y \circ y))) \circ (y \circ (x \circ x))) \circ ((0 \circ 0) \circ (0 \circ 0))] \text{ [By (S}_5)] \\
&= [(((z \circ (x \circ x)) \circ (z \circ (y \circ y))) \circ (y \circ (x \circ x))) \circ 0] \\
&\circ[(((z \circ (x \circ x)) \circ (z \circ (y \circ y))) \circ (y \circ (x \circ x))) \circ 0] \text{ [By (S}_2)] \\
&= 0. \text{ [By Lemma 3.5(2) and (S}_2)]
\end{aligned}$$

By Lemma 3.6, we get

$$((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (z \circ (y \circ y)) \leq (y \circ (x \circ x)) \circ (y \circ (x \circ x)). \tag{4.3}$$

Moreover by (S$_2$),

$$\begin{aligned}
&((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (z \circ (y \circ y)) \\
&= ((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (((z \circ (y \circ y) \circ (z \circ (y \circ y)) \circ ((z \circ (y \circ y) \circ (z \circ (y \circ y))).
\end{aligned}$$

Thus by (4.1), (4.2) and Theorem 4.4,

$$\begin{aligned}
&((z \circ (x \circ x)) \circ (z \circ (x \circ x)) \\
&\in A((((z \circ (y \circ y) \circ (z \circ (y \circ y))) \circ ((z \circ (y \circ y)) \circ (z \circ (y \circ y))), (y \circ (x \circ x)) \circ (y \circ (x \circ x))) \\
&\subset I.
\end{aligned}$$

Similarly, $(x \circ (z \circ z)) \circ (x \circ (z \circ z)) \in I$. So $x \sim_I z$. Hence $\sim_I$ is an equivalence relation on X.

Now suppose $x \sim_I u$ and $y \sim_I v$ for all $x, y, u, v \in X$. Then clearly,

$$(x \circ (y \circ u)) \circ (x \circ (u \circ u)),\ (u \circ (x \circ x)) \circ (u \circ (x \circ x)) \in I, \tag{4.4}$$

$$(y \circ (v \circ v)) \circ (y \circ (v \circ v)),\ (v \circ (y \circ y)) \circ (v \circ (y \circ y)) \in I. \tag{4.5}$$

On the other hand, we have

$$\begin{aligned}
&[(((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ (y \circ (u \circ u))) \circ ((u \circ (x \circ x)) \circ (u \circ (x \circ x)))] \\
&\circ[(((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ (y \circ (u \circ u))) \circ ((u \circ (x \circ x)) \circ (u \circ (x \circ x)))] \\
&= [(((y \circ (x \circ x)) \circ (0 \circ 0)) \circ (y \circ (u \circ u))) \circ ((u \circ (x \circ x)) \circ (0 \circ 0))] \\
&\circ[(((y \circ (x \circ x)) \circ (0 \circ 0)) \circ (y \circ (u \circ u))) \circ ((u \circ (x \circ x)) \circ (0 \circ 0))] \text{ [By (SSKU}_2)] \\
&= [(((y \circ (x \circ x)) \circ (y \circ (u \circ u))) \circ (u \circ (x \circ x))) \circ ((0 \circ 0) \circ (0 \circ 0))] \\
&\circ[(((y \circ (x \circ x)) \circ (y \circ (u \circ u))) \circ (u \circ (x \circ x))) \circ ((0 \circ 0) \circ (0 \circ 0))] \text{ [By (S}_5)]
\end{aligned}$$

$= [(((y \circ (x \circ x)) \circ (y \circ (u \circ u))) \circ (u \circ (x \circ x))) \circ 0]$
$\circ[(((y \circ (x \circ x)) \circ (y \circ (u \circ u))) \circ (u \circ (x \circ x))) \circ 0]$ [By (S_2)]
$= 0$. [By Lemma 3.5(2) and (S_2)]
By Lemma 3.6, we get

$$((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ (y \circ (u \circ u)) \leq (u \circ (x \circ x)) \circ (u \circ (x \circ x)). \tag{4.6}$$

Thus by (4.4) and Proposition 4.3,

$$((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ (y \circ (u \circ u)) \in I. \tag{4.7}$$

So we have

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)) \sim_I (y \circ (u \circ u)) \circ (y \circ (u \circ u)). \tag{4.8}$$

Similarly, we obtain

$$(y \circ (u \circ u)) \circ (y \circ (u \circ u)) \sim_I (v \circ (u \circ u)) \circ (v \circ (u \circ u)). \tag{4.9}$$

Since $\sim_I$ is transitive, by (4.8) and (4.9),

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)) \sim_I (v \circ (u \circ u)) \circ (v \circ (u \circ u)). \tag{4.10}$$

Hence $\sim_I$ is an $SSKU$-congruence on X. □

Remark 4.8. Let I be a $SSKU$-ideal of an associative $SSKU$-algebra $(X, \circ, 0)$ and $x \in X$. Then the subset C_x of X defined by:

$$C_x = \{y \in X : y \sim_I x\}$$

is called the *Sheffer stroke KU-congruence class* (briefly, $SSKU$-congruence class) induced by I and the collection $\{C_x : x \in X\}$ will be denoted by X/I. It is obvious that $x \in C_x$.

Proposition 4.9. *Let I be a $SSKU$-ideal of an associative $SSKU$-algebra $(X, \circ, 0)$. Then $I = C_0$.*

Proof. Let $x \in I$. Then by Lemma 3.5(11), (12) and ($SSKUI_1$), we have

$$(x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0)) \in I, \; (0 \circ (x \circ x)) \circ (0 \circ (x \circ x)) = 0 \in I.$$

Thus $x \sim_I 0$. So $x \in C_0$, i.e., $I \subset C_0$.

Conversely, let $x \in C_0$. Then clearly, $(x \circ (0 \circ 0)) \circ (x \circ (0 \circ 0)) \in I$. Since $0 \in I$, by ($SSKUI_2$), $x \in I$. So $C_0 \subset I$. Hence $I = C_0$. □

Proposition 4.10. *Let I be a $SSKU$-ideal of an associative $SSKU$-algebra $(X, \circ, 0)$. We define the binary operation $*$ on X/I as follows: for all $x, y \in X$,*

$$C_x * C_y = C_{(y \circ (x \circ x)) \circ (y \circ (x \circ x))}.$$

Then $(X/I, *, C_0)$ is a KU-algebra.

In this case, $(X/I, *, C_0)$ is called a *quotient KU-algebra* of X induced by I.

Proof. Since $\sim_I$ is a $SSKU$-congruence on X, $*$ is well-defined. Let $x, y, z \in X$. Then by the definition of $*$ and (S_2), we have

$$\begin{aligned}&(C_x * C_y) * [(C_y * C_z) * (C_x * C_z)] \\ =\ &C_{[((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (((z \circ (y \circ y)) \circ (y \circ (x \circ x))) \circ ((z \circ (y \circ y)) \circ (y \circ (x \circ x))))] \circ [((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \circ (((z \circ (y \circ y)) \circ (y \circ (x \circ x))) \circ ((z \circ (y \circ y)) \circ (y \circ (x \circ x))))]}\end{aligned}$$

$= C_0$. [By ($SSKU_1$)]
Thus the condition (KU_1) holds.

Now let $x \in X$. Then we get

$$C_x * C_0 = C_{(0\circ(x\circ x))\circ(0\circ(x\circ x))}$$
$$= C_0, \text{ [By Lemma 3.5(2) and } (S_2)]$$
$$C_0 * C_x = C_{(x\circ(0\circ 0))\circ(x\circ(0\circ 0))}$$
$$= C_x. \text{ [By } (SSKU_2) \text{ and } (S_2)]$$

Thus the conditions (KU_2) and (KU_3) hold.

Finally, suppose $C_x * C_y = C_0 = C_y * C_x$ for all $x, y \in X$. Then we have

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)) \sim_I (x \circ (y \circ y)) \circ (x \circ (y \circ y)).$$

Thus $C_x = C_y$. So the condition (KU_4) holds. Hence $(X/I, *, C_0)$ is a KU-algebra. □

Proposition 4.11. *Let I be a $SSKU$-deal of an associative $SSKU$-algebra $(X, \circ, 0)$. If $\pi : X \to (X/I, *, C_0)$ is the canonical mapping defined as follows: for each $x \in X$,*

$$\pi(x) = C_x,$$

then π is an $SSKU$-KU-epimorphism.

Proof. The proof is easy. □

Definition 4.12. Let $(X, \circ, 0)$ be a $SSKU$-algebra and $(Y, *, 0^{'})$ a KU-algebra. Then a mapping $f : X \to Y$ is called a *Sheffer stroke KU-KU-homorphism* (briefly, $SSKU$-KU-homomorphism), if it satisfies the following conditions: for all $x, y \in X$,

(i) $f(0) = 0^{'}$,

(ii) $f((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = f(y) * f(x)$.

The subset $ker f = f^{-1}(0^{'})$ of X is called the *Sheffer stroke KU-KU-kernel* (briefly, $SSKU$-KU-kernel) of X.

An injective [resp. surjective and bijective] $SSKU$-KU-homomorphism is called a *Sheffer stroke KU-KU-monomorphism* [resp. *epimorphism* and *isomorphism*] (briefly, $SSKU$-KU-monomorphism [resp. epimorphism and isomorphism]).

Lemma 4.13. *The $SSKU$-KU-kernel $ker f$ is an $SSKU$-ideal of X. Furthermore, $ker f$ is an $SSKU$-congruence on X such that $C_0 = ker f$, where $C_x = \{y \in X : y \sim_{ker f} x\}$ for all $x \in X$.*

Proof. From Definition 4.12(i), it is obvious that $0 \in ker f$. Then $ker f$ satisfies the condition ($SSKUI_1$). Suppose $(y \circ (x \circ x)) \circ (y \circ (x \circ x))$, $x \in ker f$. Then $f((y \circ (x \circ x)) \circ (y \circ (x \circ x))) = f(x) * f(y) = 0^{'}$. Thus by the definition of $\leq$ in a KU-algebra, $f(y) \leq f(x)$. Since $x \in ker f$, $f(x) = 0^{'}$, i.e., $f(y) \leq 0^{'}$. By ($KU_{2'}$), $f(y) = 0^{'}$. So $y \in ker f$, i.e., $ker f$ satisfies the condition ($SSKUI_2$). Hence $ker f$ is a $SSKU$-ideal of X.

The proof of the second part is similar to Propositions 4.7 and 4.9. □

Definition 4.14. Let $(X, \circ, 0)$ be an $SSKU$-algebra and A a nonempty subset of X. Then A is called a *Sheffer stroke KU-subalgebra* (briefly, $SSKU$-subalebra), if it satisfies the following condition:

$$(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in A \text{ for all } x, y \in X. \tag{4.11}$$

Definition 4.15. Let $(X, \circ, 0)$ be an $SSKU$-algebra, $(Y, *, 0^{'})$ an $SSKU$-algebra, $f : X \to Y$ a mapping, $A \subset X$ and $B \subset Y$. Then

(i) the *image of A in Y under f*, denoted by $f(A)$, is a subset of X defined by:

$$f(A) = \{f(x) \in Y : x \in X\},$$

(ii) the preimage of B in X under f, denoted by $f^{-1}(B)$, is a subset of X defined by:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Proposition 4.16. *Let $(X, \circ, 0)$ be an $SSKU$-algebra, $(Y, *, 0^{'})$ a KU-algebra and $f : X \to Y$ a $SSKU$-KU-homomorphism. Then the following hold:*

(1) *if A is an $SSKU$-subalgebra of X, then $f(A)$ is a KU-subalgebra of Y,*

(2) *if I is an $SSKU$-ideal of X, then $f(I)$ is a KU-ideal of Y,*

(3) *if B is a KU-subalgebra of Y, then $f^{-1}(B)$ is an $SSKU$-subalgebra of X,*

(4) *if J is a KU-ideal of Y, then $f^{-1}(J)$ is an $SSKU$-ideal of X,*

(5) *f is is an $SSKU$-KU-momomorphism if and only if $ker f = \{0\}$.*

Proof. (1) Suppose A is an $SSKU$-subalgebra of X and let $x, y \in f(A)$. Then there are $a, b \in X$ such that $x = f(b),\ y = f(a)$. Since f is an $SSKU$-KU-homomorphism, we have

$$x * y = f(b) * f(a) = f((a \circ (b \circ b)) \circ (a \circ (b \circ b))).$$

By the hypothesis, $(a \circ (b \circ b)) \circ (a \circ (b \circ b)) \in A$. Thus $x * y \in f(A)$. So $f(A)$ is a KU-subalgebra of Y.

(2) Suppose I is an $SSKU$-ideal of X. Then $0 \in I$. Thus by Definition 4.12(i), $0^{'} \in f(I)$. Suppose $f(a) * f(b),\ f(a) \in f(I)$ for all $a, b \in X$. Then $f(a) * f(b) = f((b \circ (a \circ a)) \circ (b \circ (a \circ a)))$. Thus $(b \circ (a \circ a)) \circ (b \circ (a \circ a)),\ a \in I$. By the hypothesis, $b \in I$, i.e., $f(b) \in f(I)$. So $f(I)$ is a KU-ideal of Y.

(3) Suppose B is a KU-subalgebra of Y and let $a, b \in f^{-1}(B)$. Then there are $x, y \in B$ such that $x = f(b),\ y = f(a)$. Thus $x * y = f(b) * f(a) = f((a \circ (b \circ b)) \circ (a \circ (b \circ b)))$. By the hypothesis, $x * y \in B$. So $f((a \circ (b \circ b)) \circ (a \circ (b \circ b))) \in B$, i.e., $(a \circ (b \circ b)) \circ (a \circ (b \circ b)) \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an $SSKU$-subalgebra of X.

(4) Suppose J is a KU-ideal of Y. Then clearly, $0^{'} \in J$. Thus by Definition 4.12(i), $0 \in f^{-1}(J)$. Now suppose $(b \circ (a \circ a)) \circ (b \circ (a \circ a)),\ a \in f^{-1}(J)$ for all $a, b \in X$. Then $f((b \circ (a \circ a)) \circ (b \circ (a \circ a)) = f(a) * f(b),\ f(a) \in J$. Since J is a KU-ideal of Y, $f(b) \in J$. Thus $b \in f^{-1}(J)$. So $f^{-1}(J)$ is an $SSKU$-ideal of X.

(5) Suppose f is is an $SSKU$-KU-momomorphism and let $x \in ker f$. Then clearly, $f(x) = 0^{'}$. Since $f(0) = 0^{'}$, $f(x) = f(0)$. Thus by the hypothesis, $x = 0$. So $ker f = \{0\}$.

Conversely, suppose $ker f = \{0\}$ and let $x, y \in X$ such that $f(x) = f(y)$. Then we have

$$f((y \circ (x \circ x)) \circ (y \circ (x \circ x))) = f(x) * f(y) = 0^{'} = f((x \circ (y \circ y)) \circ (x \circ (y \circ y))).$$

Thus we get

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)),\ (x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in ker f.$$

By the hypothesis,

$$(y \circ (x \circ x)) \circ (y \circ (x \circ x)) = 0 = (x \circ (y \circ y)) \circ (x \circ (y \circ y)).$$

So by (SSKU$_3$), $x = y$. Hence f is is an $SSKU$-KU-momomorphism. □

Proposition 4.17. *Let $(X, \circ, 0)$ be an associative SSKU-algebra and $(Y, *, 0^{'})$ a KU-algebra. If $f : X \to Y$ a SSKU-KU-epimorphism, then there is a KU-isomorphism between the quotient KU-algebra $X/kerf$ and Y.*

Proof. Suppose $f : X \to Y$ an $SSKU$-KU-epimorphism. Define the mapping $\varphi : X/kerf \to Y$ as follows: for each $x \in X$,

$$\varphi(C_x) = f(x), \text{ where } C_x = \{y \in X : y \sim_{kerf} x\} \text{ and } C_0 = kerf.$$

Let $C_x, C_y \in X/kerf$ such that $C_x = C_y$. By (2.1) and Proposition 4.10, we get

$$C_{(y\circ(x\circ x))\circ(y\circ(x\circ x))} = C_x * C_y = C_0 = C_y * C_x = C_{(x\circ(y\circ y))\circ(x\circ(y\circ y))}.$$

Then $(y \circ (x \circ x)) \circ (y \circ (x \circ x)),\ (x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in kerf$. Thus by Definition 4.12(ii), we have

$$f((y\circ(x\circ x))\circ(y\circ(x\circ x))) = f(x)*f(y) = 0^{'} = f(y)*f(x) = f((x\circ(y\circ y))\circ(x\circ(y\circ y))).$$

By (SSKU$_4$), $f(x) = f(y)$. So $\varphi(C_x) = \varphi(C_y)$. Hence φ is well-defined.

Next, suppose $C_x, C_y \in X/kerf$ such that $\varphi(C_x) = \varphi(C_y)$ for all $x, y \in X$. Then $f(x) = f(y)$. By Definition 4.12(ii) and (2.1), we have

$$f((y\circ(x\circ x))\circ(y\circ(x\circ x))) = f(x)*f(y) = 0^{'} = f(y)*f(x) = f((x\circ(y\circ y))\circ(x\circ(y\circ y))).$$

Thus $(y \circ (x \circ x)) \circ (y \circ (x \circ x)),\ (x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in Kerf$. So $x \sim_{kerf} y$, i.e., $C_x = C_y$. Hence φ is injective. It is obvious that φ is surjective.

Finally, let $C_x, C_y \in X/kerf$. Then we get

$$\begin{aligned}\varphi(C_x * C_y) &= \varphi(C_{(y\circ(x\circ x))\circ(y\circ(x\circ x))}) \text{ [By Proposition 4.10]}\\ &= f((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \text{ [By the definition of } \varphi]\\ &= f(x) * f(y) \text{ [By Definition 4.12(ii)]}\\ &= \varphi(C_x) * \varphi(C_y).\end{aligned}$$

Thus φ is a KU-homomorphism. So φ is a KU-isomorphism. □

It is obvious that if $\pi : (X, \circ, 0) \to (X/kerf, *, C_0)$ is an $SSKU$-KU-epimorphism given in Proposition 4.11, then $\varphi \circ \pi = f$.

Definition 4.18. Let $(X, \circ, 0)$ and $(Y, \circ^{'}, 0^{'})$ be $SSKU$-algebras. Then a mapping $f : X \to Y$ is called a *Sheffer stroke KU-homorphism* (briefly, $SSKU$-homomorphism), if $f(x \circ y) = f(x) \circ^{'} f(y)$ for all $x, y \in X$.

The subset $Kerf = f^{-1}(0^{'})$ of X is called the *Sheffer stroke KU-kernel* (briefly, $SSKU$-kernel) of X.

An injective [resp. surjective and bijective] $SSKU$-homomorphism is called a *Sheffer stroke KU-monomorphism* [resp. *epimorphism* and *isomorphism*] (briefly, $SSKU$-monomorphism [resp. epimorphism and isomorphism]).

Definition 4.19. Let $(X, \circ, 0)$, $(Y, \circ^{'}, 0^{'})$ be $SSKU$-algebras, $f : X \to Y$ a mapping, $A \subset X$ and $B \subset Y$. Then

(i) the *image of A in Y under f*, denoted by $f^{\to}(A)$, is a subset of X defined by:

$$f^{\to}(A) = \{f(x) \in Y : x \in X\},$$

(ii) the preimage of B in X under f, denoted by $f^{\leftarrow}(B)$, is a subset of X defined by:

$$f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}.$$

Proposition 4.20. *Let $(X, \circ, 0)$, $(Y, \circ^{'}, 0^{'})$ be SSKU-algebras be SSKU-algebras and $f : X \to Y$ an SSKU-homomorphism. Then the following hold:*
(1) $f(0) = 0^{'}$,
(2) *if A is an SSKU-subalgebra of X, then $f^{\to}(A)$ is an SSKU-subalgebra of Y,*
(3) *if I is an SSKU-ideal of X, then $f^{\to}(I)$ is an SSKU-ideal of Y,*
(4) *if B is an SSKU-subalgebra of Y, then $f^{\leftarrow}(B)$ is an SSKU-subalgebra of X,*
(5) *if J is an SSKU-ideal of Y, then $f^{\leftarrow}(J)$ is an SSKU-ideal of X,*
(6) *f is is an SSKU-momomorphism if and only if $Ker f = \{0\}$.*

Proof. (1) The proof follows from Lemma 3.5(1) and Definition 4.18.

(2) Suppose A is an $SSKU$-subalgebra of X and let $x, y \in f^{\to}(A)$. Then there are $a, b \in A$ such that $x = f(a)$ and $y = f(b)$. By the hypothesis, $(a\circ(b\circ b))\circ(a\circ(b\circ b)) \in A$. Thus $f^{\to}((a\circ(b\circ b))\circ(a\circ(b\circ b))) \in f^{\to}(A)$. Since f is an $SSKU$-homomorphism, we have

$$f^{\to}((a \circ (b \circ b)) \circ (a \circ (b \circ b))) = (x \circ^{'} (y \circ^{'} y)) \circ^{'} (x \circ^{'} (y \circ^{'} y)).$$

So $(x \circ^{'} (y \circ^{'} y)) \circ (x \circ^{'} (y \circ^{'} y)) \in f^{\to}(A)$. Hence $f^{\to}(A)$ is an $SSKU$-subalgebra of Y.

(3) Suppose I is an $SSKU$-ideal of X. Then by ($SSKUI_1$), $0 \in I$. Thus by (1), $0^{'} \in f^{\to}(I)$. So the condition ($SSKUI_1$) holds.

Now suppose $(y \circ^{'} (x \circ^{'} x)) \circ^{'} (y \circ^{'} (x \circ^{'} x))$, $x \in f^{\to}(I)$. Then there are $a \in I$ and $b \in X$ such that $x = f(a)$ and $y = f(b)$. Since f is an $SSKU$-homomorphism, we get

$$f((b \circ (a \circ a)) \circ (b \circ (a \circ a))) = (y \circ^{'} (x \circ^{'} x)) \circ^{'} (y \circ^{'} (x \circ^{'} x)) \in f^{\to}(I).$$

Thus $(b\circ(a\circ a))\circ(b\circ(a\circ a)) \in I$ and $a \in I$. By ($SSKUI_2$), $b \in I$. So $y = f(b) \in f^{\to}(I)$. Hence the condition ($SSKUI_2$) holds. Therefore $f^{\to}(A)$ is an $SSKU$-subalgebra of Y.

(4) Suppose B is an $SSKU$-subalgebra of Y and let $a, b \in X$ such that $f(a), f(b) \in B$. Then $(f(a) \circ^{'} (f(b) \circ^{'} f(b))) \circ^{'} (f(a) \circ^{'} (f(b) \circ^{'} f(b))) \in B$. Since f is an $SSKU$-homomorphism, we have

$$(f(a) \circ^{'} (f(b) \circ^{'} f(b))) \circ^{'} (f(a) \circ^{'} (f(b) \circ^{'} f(b))) = f((a \circ (b \circ b)) \circ (a \circ (b \circ b))).$$

Thus $(a \circ (b \circ b)) \circ (a \circ (b \circ b)) \in f^{\leftarrow}(B)$. So $f^{\leftarrow}(B)$ is an $SSKU$-subalgebra of X.

(5) Suppose J is an $SSKU$-ideal of Y. Then clearly, by ($SSKUI_1$), $0^{'} \in J$. Thus by (1), $0 \in f^{\leftarrow}(J)$. So the condition ($SSKUI_1$) holds.

Now suppose $(b \circ (a \circ a)) \circ (b \circ (a \circ a))$, $a \in f^{\leftarrow}(J)$ for all $x, y \in X$. Then we get

$$f((b\circ(a\circ a))\circ(b\circ(a\circ a))) = (f(b)\circ((f(a)\circ f(a)))\circ(f(b)\circ((f(a)\circ f(a))),\ f(a) \in J.$$

Thus by the hypothesis, $f(b) \in J$, i.e., $b \in f^{\leftarrow}(J)$. So the condition ($SSKUI_2$) holds. Hence $f^{\leftarrow}(J)$ is an $SSKU$-ideal of X.

(6) The proof is similar to one of Proposition 4.16(5). □

Lemma 4.21. *Let* $f : (X, \circ, 0) \to (Y, \circ^{'}, 0^{'})$ *be an* $SSKU$*-homomorphism. Then* $Kerf$ *is an* $SSKU$*-ideal of* X*.*

Proof. From Proposition 4.20(1), it is obvious that $0 \in Kerf$. Suppose the following condition hold: for all $a, b \in X$,

$$(b \circ (a \circ a)) \circ (b \circ (a \circ a)),\ a \in Kerf.$$

Then $f((b \circ (a \circ a)) \circ (b \circ (a \circ a))) = 0^{'}$ and $f(a) = 0^{'}$. Since f is an $SSKU$-homomorphism, we have

$$f((b \circ (a \circ a)) \circ (b \circ (a \circ a))) = (f(b) \circ^{'} (f(a) \circ^{'} f(a))) \circ^{'} (f(b) \circ^{'} (f(a) \circ^{'} f(a)) = 0^{'}.$$

Thus $f(b) \leq f(a) = 0^{'}$. By Proposition 3.7(13), $f(b) = 0^{'}$. So $b \in Kerf$. Hence $Kerf$ is an $SSKU$-ideal of X. □

Proposition 4.22. *Let* $f : (X, \circ, 0) \to (Y, \circ^{'}, 0^{'})$ *be an* $SSKU$*-homomorphism. We define binary relation* $\sim_{Kerf}$ *on* X *as follows: for all* $x, y \in X$*,*

$x \sim_{Kerf} y$ if and only if $(x \circ (y \circ y)) \circ (x \circ (y \circ y)),\ (y \circ (x \circ x)) \circ (y \circ (x \circ x)) \in \sim_{Kerf}$.

If X *and* Y *are associative* $SSKU$*-algebras, then* $Kerf$ *is an* $SSKU$*-congruence on* X*.*

Proof. The proof is similar to Proposition 4.7. □

We obtain the following result, which has a different structure than Proposition 4.10.

Proposition 4.23. *Let* $(X, \circ, 0)$ *and* $(Y, \circ^{'}, 0^{'})$ *be associative* $SSKU$*-algebras and* $f : X \to Y$ *an* $SSKU$*-homomorphism. We define the binary operation* $\circ^{'}$ *on* $X/Kerf$ *as follows: for all* $x, y \in X$*,*

$$C_x \circ^{'} C_y = C_{x \circ y}.$$

Then $(X/Kerf, \circ^{'}, C_0)$ *is an* $SSKU$*-algebra, where* $C_x = \{y \in X : y \sim_{Kerf} x\}$*.*

Furthermore, If $\pi : X \to (X/Kerf, \circ^{'}, C_0)$ *is the canonical mapping defined as follows: for each* $x \in X$*,*

$$\pi(x) = C_x,$$

then π *is an* $SSKU$*-epimorphism.*

In this case, $(X/Kerf, \circ^{'}, C_0)$ is called a *quotient Sheffer stroke KU-algebra* (briefly, quotient $SSKU$-algebra) of X induced by $Kerf$.

Proof. It is clear that $C_0 = Kerf$. The proof of the first part follows from Definition 3.1 and the definition of $\circ^{'}$. The proof of the second part is easy. □

Proposition 4.24. *Let* $(X, \circ, 0)$ *and* $(Y, \circ^{'}, 0^{'})$ *are associative* $SSKU$*-algebras and* $f : X \to Y$ *an* $SSKU$*-epimorphism. Then there is a mapping* $\varphi : (X/Kerf, \circ^{'}, C_0) \to (Y, \circ^{'}, 0^{'})$ *as follows: for each* $C_x \in X/Kerf$*,*

$$\varphi(C_x) = f(x)$$

such that it is an $SSKU$*-isomorphism. Furthermore,* $\varphi \circ \pi = f$*.*

Proof. The proof is similar to Proposition 4.17. □

Proposition 4.25. *Let X, Y, Z be associative $SSKU$-algebras, $f : X \to Y$ an $SSKU$-epimorphism and $g : X \to Z$ an $SSKU$-homomorphism. If $Kerf \subset Kerg$, then there is a unique $SSKU$-homomorphism $\varphi : Y \to Z$ such that $\varphi \circ f = g$.*

Proof. Let $x \in Y$. Since f is surjective, there $x_y \in X$ such that $f(x_y) = y$. Then we can define a mapping $\varphi : Y \to Z$ as follows: for each $y \in Y$,

$$\varphi(y) = g(x_y).$$

Suppose $a = b$ for all $a, b \in Y$. Since f is surjective, there are $x_a, x_b \in X$ such that $a = f(x_a)$ and $b = f(x_b)$. Then $f(x_a) = f(x_b)$. Thus by Lemma 3.5(1), we have

$$f((x_a \circ (x_b \circ x_b)) \circ (x_a \circ (x_b \circ x_b))) = (f(x_a) \circ (f(x_b) \circ f(x_b))) \circ (f(x_a) \circ (f(x_b) \circ f(x_b))) = 0,$$

$$f((x_b \circ (x_a \circ x_a)) \circ (x_b \circ (x_a \circ x_a))) = (f(x_b) \circ (f(x_a) \circ f(x_a))) \circ (f(x_b) \circ (f(x_a) \circ f(x_a))) = 0,$$

So we get

$$(x_a \circ (x_b \circ x_b)) \circ (x_a \circ (x_b \circ x_b)) \in Kerf,$$
$$(x_b \circ (x_a \circ x_a)) \circ (x_b \circ (x_a \circ x_a)) \in Kerf.$$

Since $Kerf \subset Kerg$ and g is an $SSKU$-homomorphism, we have

$$(g(x_a) \circ (g(x_b) \circ g(x_b))) \circ (g(x_a) \circ (g(x_b) \circ g(x_b))) = 0,$$
$$(g(x_b) \circ (g(x_a) \circ g(x_a))) \circ (g(x_b) \circ (g(x_a) \circ g(x_a))) = 0.$$

By (SSKU_3), $g(x_a) = g(x_b)$, i.e., $\varphi(a) = \varphi(b)$. Hence φ is well-defined.

The proofs of which $\varphi \circ f = f$, φ is an $SSKU$-homomorphism and is unique follow Theorem 3 in [30]. □

5. Conclusions

By defining the concept of KU-ideals of an $SSKU$-algebra, we obtained some of its properties. In particular, we dealt with a relationship between $SSKU$-ideals]and classical KU-ideals of a KU-algebra. Also, we discussed some properties of the images and the preimages of $KSSU$-ideals of an $SSKU$-algebra under an $SSKU$-homomorphism.

In the future, we expect to apply $SSKU$-algebra to fuzzy sets, bipolar fuzzy sets, hesitant fuzzy sets, Pythagorean fuzzy sets and neutrosophic sets etc.

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